SPLITTING THE KÜNNETH FORMULA

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Abstract. There is a description of the torsion product of two modules in terms of generators and relations given by Eilenberg and Mac Lane. With some additional data on the chain complexes there is a splitting of the map in the Künneth formula in terms of these generators. Different choices of this additional data determine a natural coset reminiscent of the indeterminacy in a Massey triple product. In one class of examples the coset actually is a Massey triple product.

The explicit formulas for a splitting enable proofs of results on the behavior of the interchange map and the long exact sequence boundary map on all the terms in the Künneth formula. Information on the failure of naturality of the splitting is also obtained.

1. Introduction

Fix a principal ideal domain R and let A_* and B_* be two chain complexes of R modules. The Künneth formula states that if $A_* *_R B_*$ is acyclic then there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(A_*) \otimes_R H_j(B_*) \xrightarrow{\times} H_n(A_* \otimes_R B_*) \xrightarrow{\beta} \bigoplus_{i+j=n-1} H_i(A_*) *_R H_j(B_*) \to 0$$

which is natural for pairs of chain maps and which is split. For a proof in this generality see for example Dold [1, VI, 9.13].

Let $\beta_{k,\ell}\colon H_n(A_* \otimes_R B_*) \to H_k(A_*) *_R H_\ell(B_*)$ denote β followed by projection. Say that a map $\sigma\colon H_i(A_*) *_R H_j(B_*) \to H_{i+j+1}(A_* \otimes_R B_*)$ *splits the Künneth formula at* (i,j) provided $\beta_{k,\ell} \circ \sigma = 1_{H_i(A_*) *_R H_j(B_*)}$ if $(k,\ell) = (i,j)$ and is 0 otherwise.

2. The main idea

Suppose the R modules in the complexes A_* and B_* are free, so the Künneth formula holds. The general case is discussed in §4.

In [?Eilenberg-MacLane, §11] Eilenberg and Mac Lane gave a generators and relations description of the torsion product: $A *_R B$ is the free R module on symbols $\langle \mathfrak{a}, r, \mathfrak{b} \rangle$ where $r \in R$, $\mathfrak{a} \in A$ with $\mathfrak{a} r = 0$ and $\mathfrak{b} \in B$ with $r\,\mathfrak{b} = 0$ modulo four types of relations described below, (2.7.1) – (2.7.4). The symbols $\langle \mathfrak{a}, r, \mathfrak{b} \rangle$ will be called *elementary tors.*

In what follows, given any complex C_*, $\mathbf{Z}_*(C_*)$ denotes the cycles and $\mathbf{B}_*(C_*)$ denotes the boundaries. Given any cycle $\hat{\mathfrak{c}}$ of degree $|\mathfrak{c}|$ in C_*, write $[\hat{\mathfrak{c}}] \in H_{|\mathfrak{c}|}(C_*)$ for the homology class $\hat{\mathfrak{c}}$ represents. Let $[\_]^C\colon \mathbf{Z}_*(C_*) \to H_*(C_*)$ denote the canonical map.

Mac Lane [5, Prop. V.10.6] describes a cycle in $H_n(A_* \otimes_R B_*)$ representing a given elementary tor in the range of β. Mac Lane's cycle is defined as follows. Lift $\mathfrak{a}$ to a cycle, $\hat{\mathfrak{a}}$, and $\mathfrak{b}$ to a cycle $\hat{\mathfrak{b}}$. Since $\mathfrak{a}r = 0$, $\hat{\mathfrak{a}}r$ is a boundary. Choose $X_{\hat{\mathfrak{a}}} \in A_{|\mathfrak{a}|+1}$ so that $\partial^A_{|\mathfrak{a}|+1}(X_{\hat{\mathfrak{a}}}) = \hat{\mathfrak{a}}r$. Choose $X_{\hat{\mathfrak{b}}}$ so that $\partial^B_{|\mathfrak{b}|+1}(X_{\hat{\mathfrak{b}}}) = r\hat{\mathfrak{b}}$. Up to sign and notation, Mac Lane's cycle is given by

$$M\big(\hat{\mathfrak{a}}, X_{\hat{\mathfrak{a}}}; \hat{\mathfrak{b}}, X_{\hat{\mathfrak{b}}}\big) = (-1)^{|\mathfrak{a}|+1}\hat{\mathfrak{a}} \otimes X_{\hat{\mathfrak{b}}} + X_{\hat{\mathfrak{a}}} \otimes \hat{\mathfrak{b}} \tag{2.1}$$

Mac Lane puts the sign in front of the other term but then gets a sign when evaluating β. Mac Lane also writes (2.1) as a Bockstein.

The short exact sequence $0 \to R \xrightarrow{r} R \xrightarrow{\rho^r} R/(r) \to 0$ gives rise to a long exact sequence whose boundary term is called the Bockstein associated to the sequence: $\mathfrak{b}^r_n\colon H_n\big(C_* \otimes_R R/(r)\big) \to H_{n-1}(C_*)$ In terms of the Bockstein and the pairing

$$H_k\big(A_* \otimes_R R/(r)\big) \times H_\ell\big(B_* \otimes_R R/(r)\big) \to H_{k+\ell}\big(A_* \otimes_R B_* \otimes_R R/(r)\big)$$

$$M\big(\hat{\mathfrak{a}}, X_{\hat{\mathfrak{a}}}; \hat{\mathfrak{b}}, X_{\hat{\mathfrak{b}}}\big) = (-1)^{|\mathfrak{a}|+1}\mathfrak{b}^r_{|\mathfrak{a}|+|\mathfrak{b}|+2}\big(X_{\hat{\mathfrak{a}}} \otimes X_{\hat{\mathfrak{b}}}\big) \tag{2.2}$$

Given a different choice of cycle for $\hat{\mathfrak{a}}$, say $\hat{\mathfrak{a}}_1$, $\hat{\mathfrak{a}}_1 = \hat{\mathfrak{a}} + \partial^A_{|\mathfrak{a}|+1}(b_{\mathfrak{a}})$. Take $X_{\hat{\mathfrak{a}}_1} = X_{\hat{\mathfrak{a}}} + b_{\mathfrak{a}}r$. With a similar choice of lift on the right, $M\big(\hat{\mathfrak{a}}_1, X_{\hat{\mathfrak{a}}_1}; \hat{\mathfrak{b}}_1, X_{\hat{\mathfrak{b}}_1}\big) - M\big(\hat{\mathfrak{a}}, X_{\hat{\mathfrak{a}}}; \hat{\mathfrak{b}}, X_{\hat{\mathfrak{b}}}\big)$ is a boundary and so different choices of cycles give the same homology class.

Indeterminacy comes from the choices of $X_{\hat{\mathfrak{a}}}$ and $X_{\hat{\mathfrak{b}}}$. With $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$ fixed, $X_{\hat{\mathfrak{a}}}$ is determined up to a cycle. Let $X_{\hat{\mathfrak{a}}_1} = X_{\hat{\mathfrak{a}}} + z_{\mathfrak{a}}$ and let $X_{\hat{\mathfrak{b}}_1} = X_{\hat{\mathfrak{b}}} + z_{\mathfrak{b}}$. Then

$$\big[M\big(\mathfrak{a}, X_{\hat{\mathfrak{a}}_1}; \mathfrak{b}, X_{\hat{\mathfrak{b}}_1}\big)\big] = \big[M\big(\mathfrak{a}, X_{\hat{\mathfrak{a}}}; \mathfrak{b}, X_{\hat{\mathfrak{b}}}\big) + (-1)^{|z_{\mathfrak{a}}|}\big(\mathfrak{a} \times [z_{\mathfrak{b}}]\big)\big] + \big([z_{\mathfrak{a}}] \times \mathfrak{b}\big)$$

Since $[z_{\mathfrak{a}}]$ and $[z_{\mathfrak{b}}]$ can be chosen arbitrarily, any element in the coset $\big(\mathfrak{a} \times H_{|\mathfrak{b}|+1}(B_*)\big) \oplus \big(H_{|\mathfrak{a}|+1}(A_*) \times \mathfrak{b}\big)$ can be realized. Let

$$\langle\!\langle \mathfrak{a}, r, \mathfrak{b} \rangle\!\rangle \subset H_{|\mathfrak{a}|+|\mathfrak{b}|+1}(A_* \otimes_R B_*) \tag{2.3}$$

denote the coset determined by any of the $\big[M\big(\mathfrak{a}, X_{\hat{\mathfrak{a}}_1}; \mathfrak{b}, X_{\hat{\mathfrak{b}}_1}\big)\big]$.

The above discussion and Proposition V.10.6 of [5] shows the following.

Lemma 2.4. *For two complexes of free R modules, R a PID, the element $\left[M\left(\mathfrak{a}, X_{\hat{\mathfrak{a}}}; \mathfrak{b}, X_{\hat{\mathfrak{b}}}\right)\right]$ determines $\langle\!\langle \mathfrak{a}, r, \mathfrak{b} \rangle\!\rangle$ a well-defined coset of $\left(\mathfrak{a} \times H_{|\mathfrak{b}|+1}(B_*)\right) \oplus \left(H_{|\mathfrak{a}|+1}(A_*) \times \mathfrak{b}\right)$ such that*

$$\beta_{s,t}\left(\langle\!\langle \mathfrak{a}, r, \mathfrak{b} \rangle\!\rangle\right) = \begin{cases} \langle \mathfrak{a}, r, \mathfrak{b} \rangle & s = |\mathfrak{a}|, t = |\mathfrak{b}| \\ 0 & \textit{otherwise} \end{cases}$$

To get a splitting requires one more step. Since R is a PID, the set of boundaries in a free chain complex is a free submodule and hence there is a splitting of the boundary maps. Choose splittings for the complexes being considered here: $s_A \colon \mathbf{B}_*(A_*) \to A_{*+1}$ and $s_B \colon \mathbf{B}_*(B_*) \to B_{*+1}$.

Define

$$M\left(\hat{\mathfrak{a}}, s_A; \hat{\mathfrak{b}}, s_B; r\right) = (-1)^{|\mathfrak{a}|+1} \hat{\mathfrak{a}} \otimes s_B\left(r\hat{\mathfrak{b}}\right) + s_A\left(\hat{\mathfrak{a}} r\right) \otimes \hat{\mathfrak{b}} \tag{2.5.1}$$

$$M\left(\hat{\mathfrak{a}}, s_A; \hat{\mathfrak{b}}, s_B; r\right) = (-1)^{|\mathfrak{a}|+1} \mathfrak{b}^r_{|\mathfrak{a}|+|\mathfrak{b}|+2}\left(s_A\left(\hat{\mathfrak{a}} r\right) \otimes s_B\left(r\hat{\mathfrak{b}}\right)\right) \tag{2.5.2}$$

Lemma 2.6. *The homology class $\left[M\left(\hat{\mathfrak{a}}, s_A; \hat{\mathfrak{b}}, s_B; r\right)\right]$ is independent of the choice of cycles $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$.*

Proof. See the paragraph just below (2.2). □

Define

$$\mu^{s_A, s_B}_{|\mathfrak{a}|,|\mathfrak{b}|}\left(\langle \mathfrak{a}, r, \mathfrak{b} \rangle\right) = \left[M\left(\hat{\mathfrak{a}}, s_A; \hat{\mathfrak{b}}, s_B; r\right)\right]$$

Theorem 2.7. *For fixed splittings s_A and s_B, the function $\mu^{s_A, s_A}_{|\mathfrak{a}|,|\mathfrak{b}|}$ defined on elementary tors induces an R module map*

$$\mu^{s_A, s_B}_{|\mathfrak{a}|,|\mathfrak{b}|} \colon H_{|\mathfrak{a}|}(A_*) *_R H_{|\mathfrak{b}|}(B_*) \to H_{|\mathfrak{a}|+|\mathfrak{b}|+1}(A_* \otimes_R B_*)$$

which splits the Künneth formula at $\left(|\mathfrak{a}|, |\mathfrak{b}|\right)$.

Proof. The splitting at $\left(|\mathfrak{a}|, |\mathfrak{b}|\right)$ follows from Lemma 2.4. Fix splittings and let $\{\mathfrak{a}, \mathfrak{b}\}_r = [M\left(\hat{\mathfrak{a}}, s_A; \hat{\mathfrak{b}}, s_B; r\right)]$. By Eilenberg and Mac Lane [`?Eilenberg-MacLane`, §11], to prove μ is a module map, it suffices to prove the following

(2.7.1) $\{\mathfrak{a}_1, \mathfrak{b}\}_r + \{\mathfrak{a}_2, \mathfrak{b}\}_r = \{\mathfrak{a}_1 + \mathfrak{a}_2, \mathfrak{b}\}_r$ $\qquad \mathfrak{a}_i r = 0;\ r\mathfrak{b} = 0$
(2.7.2) $\{\mathfrak{a}, \mathfrak{b}_1\}_r + \{\mathfrak{a}, \mathfrak{b}_2\}_r = \{\mathfrak{a}, \mathfrak{b}_1 + \mathfrak{b}_2\}_r$ $\qquad \mathfrak{a} r = 0;\ r\mathfrak{b}_i = 0$
(2.7.3) $\{\mathfrak{a}, \mathfrak{b}\}_{r_1 \cdot r_2} = \{\mathfrak{a} r_1, \mathfrak{b}\}_{r_2}$ $\qquad \mathfrak{a} r_1 r_2 = 0;\ r_2\mathfrak{b} = 0$
(2.7.4) $\{\mathfrak{a}, \mathfrak{b}\}_{r_1 \cdot r_2} = \{\mathfrak{a}, r_2\mathfrak{b}\}_{r_1}$ $\qquad \mathfrak{a} r_1 = 0;\ r_1 r_2\mathfrak{b} = 0$

These formulas are easily verified at the chain level using (2.5.1), Lemma 2.6 and carefully chosen cycles. □

Remark 2.8. Eilenberg and Mac Lane work over $\mathbb{Z}$ but, as pointed out explicitly in [`?MacLaneslides`, about the middle of page 285], the proof uses nothing more than that submodules of free modules are free and that finitely generated modules are direct sums of cyclic modules. Hence the results are valid for PID's.

Remark 2.9. The data contained in a splitting is surely related to the structure introduced by Heller in [3]. See also Section 5.

3. Free Approximations

A result attributed to Dold by Mac Lane [5, Lemma 10.5] is that given any chain complex over a PID there exists a free chain complex with a quasi-isomorphic chain map to the original complex. In this paper any such complex and quasi-isomorphism will be called a *free approximation.*

Warning. Some authors also require the chain map to be surjective.

Here is a review of a construction of a free approximation, mostly to establish notation. Some lemmas needed later are also proved here.

A *weak splitting* of a chain complex A_* at an integer n is a free resolution $0 \longrightarrow \mathcal{B}^A_n \xrightarrow{\iota^A_n} \mathcal{Z}^A_n \xrightarrow{\hat\gamma_{H_n(A_*)}} H_n(A_*) \longrightarrow 0$ and a pair of maps $\mathfrak{g}^A_n = \{\gamma^A_n, \theta^A_n\}$ of the resolution into A_* where $\gamma^A_n \colon \mathcal{Z}^A_n \to \mathbf{Z}_n(A_*)$ and $\theta^A_n \colon \mathcal{B}^A_n \to A_{n+1}$. It is further required that

$$\begin{tikzcd} \mathcal{Z}^A_n \arrow[r, "\gamma^A_n"] \arrow[dr, "\hat\gamma_{H_n(A_*)}"'] & \mathbf{Z}_n(A_*) \arrow[d, "{[\_]^A_n}"] \\ & H_n(A_*) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{B}^A_n \arrow[r, "\iota^A_n"] \arrow[d, "\theta^A_n"] & \mathcal{Z}^A_n \arrow[d, "\gamma^A_n"] \\ A_{n+1} \arrow[r, "\partial^A_{n+1}"] & \mathbf{Z}_n(A_*) \end{tikzcd} \quad \text{commute.}$$

The complex is said to be *weakly split* if it is weakly split at n for all integers n. Any module over a PID has a free resolution and any complex has a weak splitting. If the complex is free, a splitting as in §2 is a weak splitting.

Given a weakly split complex, define a complex whose groups are $\mathcal{F}^A_n = \mathcal{B}^A_{n-1} \oplus \mathcal{Z}^A_n$ and whose boundary maps are the compositions

$$\partial^{\mathcal{F}^A}_n \colon \mathcal{F}^A_n = \mathcal{B}^A_{n-1} \oplus \mathcal{Z}^A_n \longrightarrow \mathcal{B}^A_{n-1} \xrightarrow{\iota^A_{n-1}} \mathcal{Z}^A_{n-1} \longrightarrow \mathcal{B}^A_{n-2} \oplus \mathcal{Z}^A_{n-1} = \mathcal{F}^A_{n-1}$$

The submodule $0 \oplus \mathcal{B}^A_{n-1} \subset \mathcal{B}^A_{n-2} \oplus \mathcal{Z}^A_{n-1} = \mathcal{F}^A_{n-1}$ is the image of $\partial^{\mathcal{F}^A}_n$ so one choice of splitting, called the *canonical splitting*, is the composition

$$s_{\mathcal{F}^A} \colon \mathbf{B}_{n-1}(\mathcal{F}^A_*) = 0 \oplus \mathcal{B}^A_{n-1} \longrightarrow \mathcal{B}^A_{n-1} \longrightarrow \mathcal{B}^A_{n-1} \oplus \mathcal{Z}^A_n = \mathcal{F}^A_n$$

Lemma 3.1. *The map*

$$\eta_n^A = \theta_{n-1}^A + \gamma_n^A \colon \mathcal{F}_n^A = \mathcal{B}_{n-1}^A \oplus \mathcal{Z}_n^A \to A_n$$

is a chain map which is a quasi-isomorphism. If $\gamma_*^A \colon \mathcal{Z}_*^A \to \mathbf{Z}_*(A_*)$ is onto then η_*^A is onto. It is always possible to choose γ_*^A to be onto.

The proofs of the claimed results are standard.

Lemma 3.2. *Let $e_* \colon A_* \to B_*$ be a surjective chain map and let $\eta_*^B \colon F_*^B \to B_*$ be a free approximation. Then there exist free approximations $\eta_*^A \colon F_*^A \to A_*$ and surjective chain maps $h_*^e \colon F_*^A \to F_*^B$ making*

$$\begin{tikzcd} F_*^A \arrow[r, "h_*^e"] \arrow[d, "\eta_*^A"] & F_*^B \arrow[d, "\eta_*^B"] \\ A_* \arrow[r, "e_*"] & B_* \end{tikzcd}$$

commute.

Proof. Let $\begin{tikzcd} P_* \arrow[r, "\hat{e}_*"] \arrow[d, "\zeta_*"] & F_*^B \arrow[d, "\eta_*^B"] \\ A_* \arrow[r, "e_*"] & B_* \end{tikzcd}$ be a pull back. Since e_* is onto, so is $\hat{e}_*$ and the kernel complexes are isomorphic. By the 5 Lemma, ζ_* is a quasi-isomorphism. Let $\eta_*^P \colon F_*^A \to P_*$ be a surjective free approximation. Then $\eta_*^A = \zeta_* \circ \eta_*^P$ and $h_*^e = \hat{e}_* \circ \eta_*^P$ are the desired maps. □

Lemma 3.3. *If $0 \to A_* \xrightarrow{e_*} B_* \xrightarrow{f_*} C_* \to 0$ is exact, there exist free approximations making the diagram below commute.*

$$\begin{tikzcd} 0 \arrow[r] & F_*^A \arrow[r, "h_*^e"] \arrow[d, "\eta_*^A"] & F_*^B \arrow[r, "h_*^f"] \arrow[d, "\eta_*^B"] & F_*^C \arrow[r] \arrow[d, "\eta_*^C"] & 0 \\ 0 \arrow[r] & A_* \arrow[r, "e_*"] & B_* \arrow[r, "f_*"] & C_* \arrow[r] & 0 \end{tikzcd}$$

Proof. Use Lemma 3.2 to get h^f. Let F_*^A be the kernel complex, hence free. There is a unique map η_*^A making the diagram commute. By the 5 Lemma, η_*^A is a quasi-isomorphism. □

Lemma 3.4. *Suppose $A_* *_R B_*$ is acyclic. Suppose $\eta_*^A \colon F_*^A \to A_*$ and $\eta_*^B \colon F_*^B \to B_*$ are free approximations. Then so is*

$$\eta_*^A \otimes \eta_*^B \colon F_*^A \otimes_R F_*^B \to A_* \otimes_R B_*$$

Proof. The Künneth formula is natural for chain maps so

$$
\begin{array}{ccccccccc}
0 \to & \bigoplus\limits_{i+j=n} H_i(F^A_*) \otimes_R H_j(F^B_*) & \xrightarrow{\times} & H_n(F^A_* \otimes_R F^B_*) & \xrightarrow{\beta} & \bigoplus\limits_{i+j=n-1} H_i(F^A_*) *_R H_j(F^B_*) & \to 0 \\
(3.5) & \Big\downarrow {\scriptstyle \bigoplus\limits_{i+j=n} \eta^A_* \otimes \eta^B_*} & & \Big\downarrow {\scriptstyle (\eta^A \otimes \eta^B)_*} & & \Big\downarrow {\scriptstyle \bigoplus\limits_{i+j=n-1} \eta^A_* * \eta^B_*} & \\
0 \to & \bigoplus\limits_{i+j=n} H_i(A_*) \otimes_R H_j(B_*) & \xrightarrow{\times} & H_n(A_* \otimes_R B_*) & \xrightarrow{\beta} & \bigoplus\limits_{i+j=n-1} H_i(A_*) *_R H_j(B_*) & \to 0
\end{array}
$$

commutes. The left and right vertical maps are tensor and torsion products of isomorphisms and hence isomorphisms. The middle vertical map is an isomorphism by the 5 Lemma. ☐

4. The general case

With notation and hypotheses as in Lemma 3.4, applying $(\eta^A \otimes \eta^B)_*$ to the cycle in (2.5.1) gives

$$M(\mathfrak{g}^A_*, \mathfrak{g}^B_*)\big(\hat{\mathfrak{a}}, r, \hat{\mathfrak{b}}\big) = \epsilon\, \gamma^A_*(\hat{\mathfrak{a}}) \otimes \theta^B_*\big(r\hat{\mathfrak{b}}\big) + \theta^A_*\big(\hat{\mathfrak{a}}r\big) \otimes_* (\hat{\mathfrak{b}}) \tag{4.1.1}$$

where $\hat{\mathfrak{a}} \in \mathcal{Z}^A_*$ satisfies $\hat{\gamma}^A_*(\hat{\mathfrak{a}}) = \mathfrak{a}$, $\hat{\mathfrak{b}} \in \mathcal{Z}^B_*$ satisfies $\hat{\gamma}^B_*(\hat{\mathfrak{b}}) = \mathfrak{b}$ and $\epsilon = (-1)^{|\mathfrak{a}|+1}$.

In general there is no analogue to (2.5.2) because not all complexes have the necessary Bocksteins. If A_* and B_* are torsion free then the necessary Bocksteins exist and applying $(\eta^A \otimes \eta^B)_*$ to (2.5.2) gives

$$M(\mathfrak{g}^A_*, \mathfrak{g}^B_*)\big(\hat{\mathfrak{a}}, r, \hat{\mathfrak{b}}\big) =\epsilon\, \mathfrak{b}^r_{|\mathfrak{a}|+|\mathfrak{b}|+2}\Big(\theta^A_*\big(\hat{\mathfrak{a}}\,r\big) \otimes \theta^B_*\big(r\,\hat{\mathfrak{b}}\big)\Big) \tag{4.1.2}$$

Lemma 4.2. *The homology class* $\big[M(\mathfrak{g}^A_*, \mathfrak{g}^B_*)(\hat{\mathfrak{a}}, r, \hat{\mathfrak{b}})\big]$ *is independent of the lifts* $\hat{\mathfrak{a}}$ *and* $\hat{\mathfrak{b}}$.

Proof. The cycles $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$ are cycles in $\mathcal{F}^A_*$ and $\mathcal{F}^B_*$ so the result is immediate from Lemma 2.6 ☐

Theorem 4.3. *Assume* $A_* *_R B_*$ *is acyclic. For fixed weak splittings* $\mathfrak{g}^A_*$ *and* $\mathfrak{g}^B_*$ *taking the homology class of* $M(\mathfrak{g}^A_*, \mathfrak{g}^B_*)(\hat{\mathfrak{a}}, \hat{\mathfrak{b}})$ *yields a map*

$$\mu^{\mathfrak{g}^A_*, \mathfrak{g}^B_*}_{i,j} : H_i(A_*) *_R H_j(B_*) \to H_{i+j+1}(A_* \otimes_R B_*)$$

which splits the Künneth formula at (i, j).

Proof. The cycle 4.1.1 is the image of the cycle 2.5.1 and so μ is a map by Theorem 2.7. Lemma 3.4 applies and (3.5) has exact rows. The splitting result follows from Theorem 2.7. ☐

Corollary 4.4. *The map* $\mu^{\mathfrak{g}^A_*, \mathfrak{g}^B_*}_{i,j}$ *will depend on the weak splittings. For any choices of weak splittings,* $\mu^{\mathfrak{g}^A_*, \mathfrak{g}^B_*}_{i,j}\big(\langle \mathfrak{a}, r, \mathfrak{b}\rangle\big)$ *is in the same coset of* $\big(\mathfrak{a} \times H_{j+1}(B_*)\big) \oplus \big(H_{i+1}(A_*) \times \mathfrak{b}\big)$. *Denote this coset by* $\langle\!\langle \mathfrak{a}, r, \mathfrak{b} \rangle\!\rangle$.

Proof. Suppose given two weak splittings, $\mathfrak{g}_i^A = \{\gamma_i^A, \theta_i^A\}$ and $\overline{\mathfrak{g}}_i^A = \{\overline{\gamma}_i^A, \overline{\theta}_i^A\}$. Then $\overline{\gamma}_i^A - \gamma_i^A \colon \mathcal{Z}_i^A \to \mathbf{Z}_i(A_*) \to H_i(A_*)$ is trivial so $\overline{\gamma}_i^A - \gamma_i^A \colon \mathcal{Z}_i^A \to \mathbf{B}_i(A_*)$. Since $\mathcal{Z}_i^A$ is free, there exists a lift $\Psi_i \colon \mathcal{Z}_i^A \to A_{i+1}$. Next consider $\partial_{i+1}^A\big(\Psi_i - (\overline{\theta}_i^A - \theta_i^A)\big) \colon \mathcal{Z}_i^A \to A_{i+1} \xrightarrow{\partial_{i+1}^A} A_i$. This map is also trivial so there is a unique map $\mathcal{Z}_i^A \to \mathbf{Z}_{i+1}(A_*)$ and hence a unique map $\Phi_i \colon \mathcal{Z}_i^A \to \mathbf{Z}_{i+1}(A_*) \to H_{i+1}(A_*)$. Then $\mu_{i,j}^{\overline{\mathfrak{g}}_*^A, \mathfrak{g}_*^B}\big(\langle \mathfrak{a}, r, \mathfrak{b}\rangle\big) - \mu_{i,j}^{\mathfrak{g}_*^A, \mathfrak{g}_*^B}\big(\langle \mathfrak{a}, r, \mathfrak{b}\rangle\big) = (-1)^{i+1}\big[\Phi_i(\hat{\mathfrak{a}}) \times \hat{\mathfrak{b}}\big] \in H_{i+1}(A_*) \times \mathfrak{b}$. A similar calculation shows the variation in the other variable lies in $\mathfrak{a} \times H_{j+1}(B_*)$. $\square$

5. Splitting via Universal Coefficients

In the torsion free case, Formula 4.1.2 suggests another way to produce a splitting. The Universal Coefficients formula says that for a torsion-free complex C_*, there exists a natural short exact sequence which is unnaturally split:

$$0 \longrightarrow H_n\big(C_*\big) \otimes_R R/(q) \longrightarrow H_n\big(C_* \otimes_R R/(q)\big) \xrightarrow{U_n^{C_*,q}} {}_qH_{n-1}(C_*) \longrightarrow 0$$

where for a fixed r in a PID R and an R module P, ${}_rP = P *_R R/(r)$ denotes the submodule of elements annihilated by r.
The Bockstein $\mathfrak{b}_n^q$ is the composition

$$H_n\big(C_* \otimes_R R/(q)\big) \xrightarrow{U_n^{C_*,q}} {}_qH_{n-1}(C_*) \subset H_{n-1}(C_*)$$

Theorem 5.1. *Let A_* and B_* be torsion-free complexes. Given $\mathfrak{a} \in H_i(A_*)$ pick $\overline{\mathfrak{a}} \in H_{i+1}\big(A_* \otimes_R R/(r)\big)$ such that $U_{i+1}^{A_*,r}(\overline{\mathfrak{a}}) = \mathfrak{a}$. Given $\mathfrak{b} \in H_j(B_*)$ pick $\overline{\mathfrak{b}} \in H_{j+1}\big(B_* \otimes_R R/(r)\big)$ such that $U_{j+1}^{B_*,r}(\overline{\mathfrak{b}}) = \mathfrak{b}$. On elementary tors $\langle \mathfrak{a}, r, \mathfrak{b}\rangle$ define*

$$S_{i,j}\big(\langle \mathfrak{a}, r, \mathfrak{b}\rangle\big) = (-1)^{i+1}\mathfrak{b}_{i+j+2}^r\big(\overline{\mathfrak{a}} \otimes \overline{\mathfrak{b}}\big)$$

Then $S_{i,j}\big(\langle \mathfrak{a}, r, \mathfrak{b}\rangle\big) \in \langle\!\langle \mathfrak{a}, r, \mathfrak{b}\rangle\!\rangle$.

Proof. From Corollary 4.4, $(-1)^{i+1}\mathfrak{b}_{i+j+2}^r(\overline{\mathfrak{a}} \otimes \overline{\mathfrak{b}})$ lies in $\langle\!\langle \mathfrak{a}, r, \mathfrak{b}\rangle\!\rangle$ if the splittings used are ones from a weak splitting. Any other choice of splitting for A_* is of the form $\overline{\mathfrak{a}} + X_{\mathfrak{a}}$ for $X_{\mathfrak{a}} \in H_{i+1}(A_*)$ and any other choice of splitting for B_* is of the form $\overline{\mathfrak{b}} + X_{\mathfrak{b}}$ for $X_{\mathfrak{b}} \in H_{j+1}(B_*)$. Then

$$\mathfrak{b}_{i+j+2}^r\big((\overline{\mathfrak{a}} + X_{\mathfrak{a}}) \otimes (\overline{\mathfrak{b}} + X_{\mathfrak{b}})\big) = \mathfrak{b}_{i+j+2}^r(\overline{\mathfrak{a}} \otimes \overline{\mathfrak{b}}) + X_{\mathfrak{a}} \times \mathfrak{b} + (-1)^{i+1}\mathfrak{a} \times X_{\mathfrak{b}}$$

The result follows. $\square$

If the Universal Coefficients splittings are chosen arbitrarily the map on the elementary tors may not descend to a map on the torsion product. This problem is overcome as follows. A family of splittings

$$s_n^{A,r}\colon {}_rH_n(A_*) \to H_{n+1}\big(A_* \otimes_R R/(r)\big)$$

one for each non-zero $r \in R$ is *a compatible family of splittings of A_* at n* provided, for all non-zero elements q_1, $q_2 \in R$ the diagram

$$\begin{tikzcd}
{}_{q_2}H_n(A_*) \arrow[r, "\subset"] \arrow[d, "s_n^{A,q_2}"] & {}_{q_1 q_2}H_n(A_*) \arrow[r, "\cdot q_2"] \arrow[d, "s_n^{A,q_1 q_2}"] & {}_{q_1}H_n(A_*) \arrow[d, "s_n^{A,q_1}"] \\
H_{n+1}\big(A_* \otimes_R R/(q_2)\big) \arrow[r, "q_1\cdot"] \arrow[d, "U_{n+1}^{A_*,q_2}"] & H_{n+1}\big(A_* \otimes_R R/(q_1 q_2)\big) \arrow[r, "\rho^{q_1}"] \arrow[d, "U_{n+1}^{A_*,q_1\cdot q_2}"] & H_{n+1}\big(A_* \otimes_R R/(q_1)\big) \arrow[d, "U_{n+1}^{A_*,q_1}"] \\
{}_{q_2}H_n(A_*) \arrow[r, "\subset"] & {}_{q_1 q_2}H_n(A_*) \arrow[r, "\cdot q_2"] & {}_{q_1}H_n(A_*)
\end{tikzcd}$$

commutes, where the horizontal maps are induced from the short exact sequence of modules $0 \to R/(q_2) \xrightarrow{q_1\cdot} R/(q_1 q_2) \xrightarrow{\rho^{q_1}} R/(q_1) \to 0$ and the rows are exact. The diagram consisting of the bottom two rows always commutes and the vertical maps from the first row to the third are the identity.

If the splittings come from a weak splitting of A_* then they are compatible for any n.

Theorem 5.2. *Suppose A_* and B_* are torsion-free. Given a compatible family of splittings of A_* at i and a compatible family of splittings of B_* at j, the formula*

$$S_{i,j}^{\{s_i^{A,r}(\mathfrak{a}),s_j^{B,r}(\mathfrak{b})\}}\big(\langle \mathfrak{a}, r, \mathfrak{b}\rangle\big) = (-1)^{i+1}\mathfrak{b}_{i+j+2}^{r}\big(s_i^{A,r}(\mathfrak{a}) \times s_j^{B,r}(\mathfrak{b})\big)$$

defines a map from $H_i(A_*) *_R H_j(B_*)$ to $H_{i+j+1}(A_* \otimes_R B_*)$ splitting the Künneth formula at (i, j).

Proof. It follows from Theorem 5.1 that if $S_{i,j}$ is a map then it splits the Künneth formula at (i, j).

To show $S_{i,j}^{\{s_i^{A,r}(\mathfrak{a}),s_j^{B,r}(\mathfrak{b})\}}$ is a map, it suffices to show that (2.7.1-2.7.4) hold. Equations (2.7.1) and (2.7.2) hold whether the splittings are compatible or not since the cross product, and hence $S_{i,j}^{\{s_i^{A,r}(\mathfrak{a}),s_j^{B,r}(\mathfrak{b})\}}$ is bilinear.

To verify (2.7.3) it suffices to show

$$\mathfrak{b}_{i+j+2}^{r_1 r_2}\big(s_i^{A,r_1 r_2}(\mathfrak{a}) \times s_j^{B,r_1 r_2}(\mathfrak{b})\big) = \mathfrak{b}_{i+j+2}^{r_2}\big(s_i^{A,r_2}(\mathfrak{a} r_1) \times s_j^{B,r_2}(\mathfrak{b})\big) \tag{5.3}$$

To compute a Bockstein of a homology class, $\mathfrak{c} \in H_n\big(C_* \otimes_R R/(q)\big)$, first lift to a chain, $\hat{\mathfrak{c}} \in C_n$ and then $\partial_n^C(\hat{\mathfrak{c}}) = qZ$. The class Z is unique

because C_n is torsion-free and $\mathfrak{b}_n^q(\mathfrak{c})$ is the homology class represented by Z.

There are four homology classes in (5.3). For uniform notation, given $s_n^{C,q}(\mathfrak{c})$, let $\hat{s}_n^{C,q}(\mathfrak{c})$ be a lift to a representing chain. The cross product of homology classes is represented by the tensor product of chains so $C_1 = \hat{s}_i^{A,r_1 r_2}(\mathfrak{a}) \otimes \hat{s}_j^{B,r_1 r_2}(\mathfrak{b})$ is a chain to compute the left hand side of (5.3) and $C_2 = \hat{s}_i^{A,r_2}(\mathfrak{a} r_1) \otimes \hat{s}_j^{B,r_2}(\mathfrak{b})$ is a chain to compute the right hand side of (5.3).

Note $\left[\partial_i^A\big(\hat{s}_i^{A,r_1 r_2}(\mathfrak{a})\big)\right] = s_i^{A,r_1 r_2}(\mathfrak{a})(r_1 r_2)$ and $\left[\partial_i^A\big(\hat{s}_i^{A,r_2}(\mathfrak{a} r_1)\big)\right] = s_i^{A,r_2}(\mathfrak{a} r_1)(r_1 r_2)$. If the splittings are compatible, $s_i^{A,r_1 r_2}(\mathfrak{a}) = s_i^{A,r_2}(\mathfrak{a} r_1)$ so choose $\hat{s}_i^{A,r_1 r_2}(\mathfrak{a}) = \hat{s}_i^{A,r_2}(\mathfrak{a} r_1)$.

Also $\left[\partial_i^B\big(\hat{s}_i^{B,r_1 r_2}(\mathfrak{b})\big)\right] = (r_1 r_2) s_i^{B,r_1 r_2}(\mathfrak{b})$ whereas $\left[\partial_i^B\big(\hat{s}_i^{B,r_2}(\mathfrak{b})\big)\right] = r_2 s_i^{B,r_2}(\mathfrak{b})$. If the splittings are compatible, $s_i^{B,r_1 r_2}(\mathfrak{b}) = s_i^{B,r_2}(\mathfrak{b})$ so choose $\hat{s}_i^{B,r_1 r_2}(\mathfrak{b}) = r_1 \hat{s}_i^{B,r_2}(\mathfrak{b})$.

It follows that $C_1 = r_1 C_2$. Since

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "r_2"] \arrow[d, "1_R"'] & R \arrow[r, "\rho^{r_2}"] \arrow[d, "r_1\cdot"] & R/(r_2) \arrow[r] \arrow[d, "r_1\cdot"] & 0 \\
0 \arrow[r] & R \arrow[r, "r_1\cdot r_2"] & R \arrow[r, "\rho^{r_1\cdot r_2}"] & R/(r_1 r_2) \arrow[r] & 0
\end{tikzcd}$$

commutes, $\mathfrak{b}_{i+j+2}^{r_1 r_2}(C_1) = \mathfrak{b}_{i+j+2}^{r_2}(r_1 C_2)$ as required. □

6. Naturality of the splitting

Fix a chain map $e_*\colon A_* \to C_*$ between two weakly split chain maps. Pick a map $\mathcal{Z}_n^{e_*}\colon \mathcal{Z}_n^A \to \mathcal{Z}_n^C$ satisfying

$$[\_]_n^C \circ \gamma_n^C \circ \mathcal{Z}_n^{e_*} = e_n \circ [\_]_n^A \circ \gamma_n^A \colon \mathcal{Z}_n^A \to H_n(C_*) \tag{6.1}$$

Since the right hand square in the diagram below commutes

$$\begin{tikzcd}
\mathcal{B}_n^A \arrow[r, phantom, "\subset"] \arrow[d, dotted, "\mathcal{B}_n^{e_*}"] & \mathcal{Z}_n^A \arrow[r, "{[_]_n^A\circ\gamma_n^A}"] \arrow[d, "\mathcal{Z}_n^{e_*}"] & H_n(A_*) \arrow[d, "e_n"] \\
\mathcal{B}_n^C \arrow[r, phantom, "\subset"] & \mathcal{Z}_n^C \arrow[r, "{[_]_n^C\circ\gamma_n^C}"] & H_n(C_*)
\end{tikzcd}$$

there exists a unique map $\mathcal{B}_n^{e_*}\colon \mathcal{B}_n^A \to \mathcal{B}_n^C$ making the left hand square commute. The set of choices for $\mathcal{Z}_n^{e_*}$ consists of any one choice plus any map $L_n\colon \mathcal{Z}_*^A \to \mathcal{B}_n^C$. The restricted map is $\mathcal{B}_n^{e_*}$ plus the restriction of L_n.

The maps $\gamma_n^C \circ \mathcal{Z}_n^{e_*}$ and $e_n \circ \gamma_n^A$ have domain $\mathcal{Z}_n^A$ and range $\mathbf{Z}_n(C_*)$ and they represent the same homology class. Hence $\gamma_n^C \circ \mathcal{Z}_n^{e_*} - e_n \circ \gamma_n^A$ lands in $\mathbf{B}_n(C_*)$. Since $\mathcal{Z}_n^A$ is free, there is a lift of this difference to a map $\Psi_n^{e_*} \colon \mathcal{Z}_n^A \to C_{n+1}$ satisfying

$$\partial_{n+1}^C \circ \Psi_n^{e_*} = \gamma_n^C \circ \mathcal{Z}_n^{e_*} - e_n \circ \gamma_n^A \tag{6.2}$$

If $\mathcal{Z}_n^{e_*}$ is replaced by $\mathcal{Z}_n^{e_*} + L_n$, a choice for the new $\Psi_n^{e_*}$ is $\Psi_n^{e_*} + \theta^C \circ L_n$.

The set of solutions to (6.2) consists of one solution, $\Psi_n^{e_*}$, plus any map of the form $\Lambda_n \colon \mathcal{Z}_n^A \to \mathbf{Z}_{n+1}(C_*) \subset C_{n+1}$.

Given a fixed solution to (6.2) consider

$$\xi = \Psi_n^{e_*}\big|_{\mathcal{B}_n^A} - \left(\theta_n^C \circ \mathcal{B}_n^{e_*} - e_{n+1} \circ \theta_n^A\right) \colon \mathcal{B}_n^A \to C_{n+1}$$

Notice if $\mathcal{Z}_n^{e_*}$ is replaced by $\mathcal{Z}_n^{e_*} + L_n$, the new ξ is the same map as the old ξ. The image of ξ is contained in the cycles of C_{n+1} and so gives a map

$$\Phi_n^{e_*} = \left(\theta_n^C \circ \mathcal{B}_n^{e_*} - e_{n+1} \circ \theta_n^A\right) - \Psi_n^{e_*}\big|_{\mathcal{B}_n^A} \colon \mathcal{B}_n^A \to H_{n+1}(C_*) \tag{6.3}$$

which does not depend on the choice of $\mathcal{Z}_*^{e_*}$.

The map $\Phi_*^{e_*}$ induces a map

$$U\langle r\rangle_n^{e_*} \colon {}_rH_n(A_*) \to H_{n+1}(C_*) \otimes R/(r)$$

defined as follows. Given $\mathfrak{a} \in {}_rH_n(A_*)$ pick $\hat{\mathfrak{a}} \in \mathcal{Z}_n^A$ so that $[\gamma_n^A(\hat{\mathfrak{a}})] = \mathfrak{a}$. Then $\hat{\mathfrak{a}}r \in \mathcal{B}_n^A$ so let $U\langle r\rangle_n^{e_*}(\mathfrak{a})$ be the homology class represented by $\Phi_n^{e_*}(\hat{\mathfrak{a}}r)$ reduced mod r.

Proposition 6.4. *Given a chain map $e_* \colon A_* \to C_*$ between two weakly split chain complexes over a PID R, the map*

$$U\langle r\rangle_n^{e_*} \colon {}_rH_n(A_*) \to H_{n+1}(C_*) \otimes R/(r)$$

is well-defined regardless of the choices made in (6.1) and (6.2).

Proof. Any other choice of element in $\mathcal{Z}_n^A$ has the form $\hat{\mathfrak{a}} + b$ for $b \in \mathcal{B}_n^A$. Then $\Phi_n^{e_*}\Big(\big(\hat{\mathfrak{a}} + b\big)r\Big) = \Phi_n^{e_*}(\hat{\mathfrak{a}}r) + \Phi_n^{e_*}\big(br\big) = \Phi_n^{e_*}(\hat{\mathfrak{a}}r) + \Phi_n^{e_*}\big(b\big)r$ since $b \in \mathcal{B}_n^A$. Hence $\Phi_n^{e_*}\Big(\big(\hat{\mathfrak{a}} + b\big)r\Big)$ and $\Phi_n^{e_*}(\hat{\mathfrak{a}}r)$ represent the same element in $H_{n+1}(C_*) \otimes R/(r)$ and therefore $U\langle r\rangle_n^{e_*}$ is well-define. Since $\Phi_n^{e_*}$ is an R module map, so is $U\langle r\rangle_n^{e_*}$.

Given a second lift, it has the form $\Psi_n^{e_*} + \Lambda$ where $\Lambda \colon \mathcal{Z}_n^A \to \mathbf{Z}_{n+1}(C_*)$ and the new Φ_* is $\Phi_n^{e_*} - \Lambda$. Compute $\big(\Phi_n^{e_*} - \Lambda\big)(\hat{\mathfrak{a}}r) = \Phi_n^{e_*}(\hat{\mathfrak{a}}r) - \Lambda(\hat{\mathfrak{a}}r)$ But Λ is defined on all of $\mathcal{Z}_n^A$ so $\big(\Phi_n^{e_*} - \Lambda\big)(\hat{\mathfrak{a}}r) = \Phi_n^{e_*}(\hat{\mathfrak{a}}r) - \Lambda(\hat{\mathfrak{a}})r$ and $U\langle r\rangle_n^{e_*}$ is independent of the lift. $\square$

Remark. A similar result holds for left R modules.

Definition 6.5. A *weak split chain map* between two weakly split chain complexes $\{A_*, \mathfrak{g}_*^A\}$ and $\{C_*, \mathfrak{g}_*^C\}$ consists of a chain map $e_*\colon A_* \to C_*$, a map $\mathcal{Z}_*^{e_*}\colon \mathcal{Z}_*^A \to \mathcal{Z}_*^C$ satisfying (6.1) and a map $\Psi_n^{e_*}\colon \mathcal{Z}_n^A \to C_{n+1}$ satisfying (6.2). From the above discussion, given any two weakly split chain complexes and a chain map between them, this data can be completed to a weakly split chain map. The map $U\langle r\rangle_*^{e_*}$ is independent of this completion.

Theorem 6.6. *Suppose given four weakly split complexes and weakly split chain maps $e_*\colon A_* \to C_*$ and $f_*\colon B_* \to D_*$.*

If $\langle \mathfrak{a}, r, \mathfrak{b}\rangle \in H_i(A_*) *_R H_j(B_*)$ then

$$\mu_{i,j}^{\mathfrak{g}_*^C,\mathfrak{g}_*^D}\big(\langle e_*(\mathfrak{a}), r, f_*(\mathfrak{b})\rangle\big) = \big(e_* \otimes f_*\big)_*\big(\mu_{i,j}^{\mathfrak{g}_*^A,\mathfrak{g}_*^B}(\langle \mathfrak{a}, r, \mathfrak{b}\rangle)\big) +$$
$$(-1)^i e_*(\mathfrak{a}) \times U\langle r\rangle_j^{f_*}(\mathfrak{b}) + U\langle r\rangle_i^{e_*}(\mathfrak{a}) \times f_*(\mathfrak{b})$$

Remark 6.7. The $U\langle r\rangle_*$ maps take values in $H_*(\_) \otimes R/(r)$ but since the other factor in the cross product is r-torsion, each cross product is well-defined in $H_{i+j+1}(C_* \otimes_R D_*)$.

Proof. It suffices to check the formula on elementary tors so fix $\langle \mathfrak{a}, r, \mathfrak{b}\rangle$. The corresponding cycle 4.1 is

$$X_0 = (-1)^{|\mathfrak{a}|+1}\gamma_i^A(\hat{\mathfrak{a}}) \otimes \theta_j^B\big(r\hat{\mathfrak{b}}\big) + \theta_i^A\big(\hat{\mathfrak{a}}r\big) \otimes \gamma_j^B(\hat{\mathfrak{b}})$$

Evaluating $e_* \otimes f_*$ on X_0 gives

$$X_1 = (-1)^{i+1} e_i\big(\gamma_i^A(\hat{\mathfrak{a}})\big) \otimes f_{j+1}\Big(\theta_j^B\big(r\hat{\mathfrak{b}}\big)\Big) + e_{i+1}\Big(\theta_i^A\big(\hat{\mathfrak{a}}r\big)\Big) \otimes f_j\big(\gamma_j^B(\hat{\mathfrak{b}})\big)$$

and a chain representing $\mu_{i,j}^{\mathfrak{g}_*^C,\mathfrak{g}_*^D}\big(\langle e_*(\mathfrak{a}), r, f_*(\mathfrak{b})\rangle\big)$ is

$$X_2 = (-1)^{i+1}\gamma_i^C\big(\mathcal{Z}_i^{e_*}(\hat{\mathfrak{a}})\big) \otimes \Big(\theta_j^D\big(r\mathcal{Z}_j^{f_*}(\hat{\mathfrak{b}})\big)\Big) + \Big(\theta_j^C\big(\mathcal{Z}_j^{e_*}(\hat{\mathfrak{a}})r\big)\Big) \otimes \gamma_j^D\big(\mathcal{Z}_j^{f_*}(\hat{\mathfrak{b}})\big)$$

It suffices to prove the theorem for $e_* \otimes 1_{B_*}$ and then for $1_{C_*} \otimes f_*$ and these calculations are straightforward. ☐

Corollary 6.8. *Given chain maps $e_*\colon A_* \to C_*$ and $f_*\colon B_* \to D_*$*

$$\big(e_* \otimes f_*\big)_*\big(\langle\!\langle \mathfrak{a}, r, \mathfrak{b}\rangle\!\rangle\big) \subset \langle\!\langle e_*(\mathfrak{a}), r, f_*(\mathfrak{b})\rangle\!\rangle$$

In words, the cosets are natural and do not depend on the weak splittings of the complexes.

Proof. First check that the 0-cosets behave correctly:
$(e_* \otimes f_*)_*\Big(\big(\mathfrak{a}\times H_{j+1}(B_*)\big)\oplus\big(H_{i+1}(A_*)\times\mathfrak{b}\big)\Big)\subset\big(e_*(\mathfrak{a})\times H_{j+1}(D_*)\big)\oplus\big(H_{i+1}(C_*)\times f_*(\mathfrak{b})\big)$
By Theorem 6.6 $\mu_{i,j}^{\mathfrak{g}_*^C,\mathfrak{g}_*^D}\big(\langle e_*(\mathfrak{a}),r,f_*(\mathfrak{b})\rangle\big)\subset\langle\!\langle e_*(\mathfrak{a}),r,f_*(\mathfrak{b})\rangle\!\rangle$. One application of Theorem 6.6 is to the case in which e_* is the identity but the weak splittings change. Hence changing the weak splittings does not change the cosets. The result follows. □

7. The interchange map and the Künneth formula

There are natural isomorphisms $I\colon A\otimes_R B\cong B\otimes_R A$ and $I\colon A *_R B\cong B *_R A$. On elementary tensors, $I(\mathfrak{a}\otimes\mathfrak{b})=\mathfrak{b}\otimes\mathfrak{a}$ and $I(\langle\mathfrak{a},r,\mathfrak{b}\rangle)=\langle\mathfrak{b},r,\mathfrak{a}\rangle$. Applying I to the tensor product of two chain complexes is not a chain map: a sign is required. The usual choice is

$$T\colon A_*\otimes_R B_*\to B_*\otimes_R A_*$$

defined on elementary tensors by $T(\mathfrak{a}\otimes\mathfrak{b})=(-1)^{|\mathfrak{a}||\mathfrak{b}|}\mathfrak{b}\otimes\mathfrak{a}$. It follows that the cross product map satisfies

$$T_*(\mathfrak{a}\times\mathfrak{b})=(-1)^{|\mathfrak{a}||\mathfrak{b}|}\mathfrak{b}\times\mathfrak{a}$$

for all $\mathfrak{a}\in H_{|\mathfrak{a}|}(A_*)$ and $\mathfrak{b}\in H_{|\mathfrak{b}|}(B_*)$.

Theorem 7.1. *For all $\mathfrak{a}\in H_i(A_*)$ and $\mathfrak{b}\in H_j(B_*)$*

$$T_*\Big(\mu_{i+j+1}^{\mathfrak{g}^A,\mathfrak{g}^B}\big(\langle\mathfrak{a},r,\mathfrak{b}\rangle\big)\Big)=(-1)^{i\cdot j+1}\mu_{i+j+1}^{\mathfrak{g}^B,\mathfrak{g}^A}\big(\langle\mathfrak{b},r,\mathfrak{a}\rangle\big)$$

Proof. Apply T to the cycle in 4.1.1. □

Corollary 7.2. *If R is a PID and if $A_* *_R B_*$ is acyclic*

$$\begin{array}{ccccccccc}
0\to & \bigoplus\limits_{i+j=n} H_i(A_*)\otimes_R H_j(B_*) & \xrightarrow{\times} & H_n(A_*\otimes_R B_*) & \xrightarrow{\beta} & \bigoplus\limits_{i+j=n-1} H_i(A_*) *_R H_j(B_*) & \to 0\\
& \Big\downarrow \bigoplus\limits_{i+j=n}(-1)^{ij}I & & \Big\downarrow T_* & & \bigoplus\limits_{i+j=n-1}(-1)^{ij+1}I\Big\downarrow & \\
0\to & \bigoplus\limits_{i+j=n} H_i(B_*)\otimes_R H_j(A_*) & \xrightarrow{\times} & H_n(B_*\otimes_R A_*) & \xrightarrow{\beta} & \bigoplus\limits_{i+j=n-1} H_i(B_*) *_R H_j(A_*) & \to 0
\end{array}$$

commutes. The splittings can be chosen to make the diagram commute.

8. The boundary map and the Künneth formula

The boundary map in question is the map associated with the long exact homology sequence for a short exact sequence of chain complexes. Before stating the result some preliminaries are needed.

Definition 8.1. A pair of composable chain maps $A_*\xrightarrow{e_*}B_*$ and $B_*\xrightarrow{f_*}C_*$ form *a weak exact sequence* provided there exists a short

exact sequence of free approximations and chain maps making (8.2) below commute.

$$\begin{array}{ccccccccc} 0 & \to & F_*^A & \xrightarrow{h_*^e} & F_*^B & \xrightarrow{h_*^f} & F_*^C & \to & 0 \\ & & \downarrow{\scriptstyle \eta_*^A} & & \downarrow{\scriptstyle \eta_*^B} & & \downarrow{\scriptstyle \eta_*^C} & & \\ & & A_* & \xrightarrow{e_*} & B_* & \xrightarrow{f_*} & C_* & & \end{array} \tag{8.2}$$

Given a weak exact sequence there is a long exact homology sequence coming from the long exact sequence of the top row of (8.2):

$$\cdots \to H_{i+1}(C_*) \xrightarrow{\boldsymbol{\partial}_{i+1}} H_i(A_*) \xrightarrow{e_*} H_i(B_*) \xrightarrow{f_*} H_i(C_*) \xrightarrow{\boldsymbol{\partial}_i} \cdots$$

The boundary $\boldsymbol{\partial}_{i+1} = \eta_*^A \circ \partial_{i+1} \circ (\eta_*^C)^{-1}$ where ∂_{i+1} is the usual boundary in the long exact homology sequence for the free complexes.

Lemma 8.3. *A short exact sequence of chain complexes*

$$0 \to A_* \xrightarrow{e_*} B_* \xrightarrow{f_*} C_* \to 0$$

is weak exact. The boundary $\boldsymbol{\partial}_{i+1}$ is the usual boundary map.

Proof. The commutative diagram of free approximations (8.2) is given by Lemma 3.3. The description of the boundary map is immediate. ☐

Lemma 8.4. *If $A_* *_R D_*$, $B_* *_R D_*$ and $C_* *_R D_*$ are acyclic and if $A_* \xrightarrow{e_*} B_* \xrightarrow{f_*} C_*$ is weak exact, then so are*

$$A_* \otimes_R D_* \xrightarrow{e_* \otimes 1_{D_*}} B_* \otimes_R D_* \xrightarrow{f_* \otimes 1_{D_*}} C_* \otimes_R D_*$$

$$D_* \otimes_R A_* \xrightarrow{1_{D_*} \otimes e_*} D_* \otimes_R B_* \xrightarrow{1_{D_*} \otimes f_*} D_* \otimes_R C_*$$

Proof. Pick free approximations satisfying (8.2), η_*^A, η_*^B, η_*^C and a free approximation η_*^D. By Lemma 3.4 the required free approximations are $\eta_*^A \otimes \eta_*^D$, $\eta_*^B \otimes \eta_*^D$, $\eta_*^C \otimes \eta_*^D$, or $\eta_*^D \otimes \eta_*^A$, $\eta_*^D \otimes \eta_*^B$, $\eta_*^D \otimes \eta_*^C$. ☐

Warning. Even if $A_* \xrightarrow{e_*} B_* \xrightarrow{f_*} C_*$ is short exact, the pair $e_* \otimes 1_{D_*}$ and $f_* \otimes 1_{D_*}$ may only be weak exact. For them to be short exact requires that either C_* or D_* be torsion free.

Theorem 8.5. *Suppose $A_* *_R D_*$, $B_* *_R D_*$ and $C_* *_R D_*$ are acyclic and suppose $A_* \xrightarrow{e_*} B_* \xrightarrow{f_*} C_*$ is weak exact. Then for $\mathfrak{a} \in H_i(C_*)$ and $\mathfrak{b} \in H_j(D_*)$*

$$\boldsymbol{\partial}_{i+j+1}\big(\langle\!\langle \mathfrak{a}, r, \mathfrak{b} \rangle\!\rangle\big) \subset -\langle\!\langle \boldsymbol{\partial}_i(\mathfrak{a}), r, \mathfrak{b} \rangle\!\rangle$$

Proof. By Lemma 8.4 it may be assumed that the complexes are all free. Pick compatible splittings for A_*, C_* and D_*. Recall that Bocksteins and long exact sequence boundary maps anti-commute and that in short exact sequences of free chain complexes $\boldsymbol{\partial}_{i+j}(\hat{\mathfrak{a}} \otimes \hat{\mathfrak{c}}) = \boldsymbol{\partial}_i(\hat{\mathfrak{a}}) \otimes \hat{\mathfrak{c}}$. A routine calculation completes the proof. □

Corollary 8.6. *With assumptions and notation as in Theorem 8.5*

$$\boldsymbol{\partial}_{i+j+1}\big(\langle\!\langle \mathfrak{b}, r, \mathfrak{a} \rangle\!\rangle\big) \subset (-1)^{j+1} \langle\!\langle \mathfrak{b}, r, \boldsymbol{\partial}_i(\mathfrak{a}) \rangle\!\rangle$$

Proof. Apply the interchange map (7.1) to get to the situation of Theorem 8.5 and then apply the interchange map again. □

Corollary 8.7. *With assumptions and notation as in Theorem 8.5 let* $\boldsymbol{\partial}_i * 1_{H_j(D_*)} \colon H_i(C_*) *_R H_j(D_*) \to H_{i-1}(A_*) *_R H_j(D_*)$ *be the map defined by* $\boldsymbol{\partial}_i * 1_{H_j(D_*)}\big(\langle \mathfrak{a}, r, \mathfrak{b} \rangle\big) = \langle \boldsymbol{\partial}_i(\mathfrak{a}), r, \mathfrak{b} \rangle$. *Then*

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{i+j=n+1} H_i(C_*) \otimes_R H_j(D_*) \arrow[r, "\times"] \arrow[d, "\bigoplus\limits_{i+j=n+1} \boldsymbol{\partial}_i \otimes 1_{H_j(D_*)}"] & H_{n+1}(C_* \otimes_R D_*) \arrow[r, "\beta"] \arrow[d, "\boldsymbol{\partial}_{n+1}"'] & \bigoplus\limits_{i+j=n} H_i(C_*) *_R H_j(D_*) \arrow[r] \arrow[d, "\bigoplus\limits_{i+j=n} -\boldsymbol{\partial}_i * 1_{H_j(D_*)}"'] & 0 \\
0 \arrow[r] & \bigoplus\limits_{i+j=n+1} H_{i-1}(A_*) \otimes_R H_j(D_*) \arrow[r, "\times"] & H_n(A_* \otimes_R D_*) \arrow[r, "\beta"] & \bigoplus\limits_{i+j=n} H_{i-1}(A_*) *_R H_j(D_*) \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof. The proof is immediate. □

9. The Massey triple product

Suppose X and Y are CW complexes with finitely many cells in each dimension. Then the cellular cochains are free $\mathbb{Z}$ modules and the Künneth formula plus the Eilenberg-Zilber chain homotopy equivalence yields a Künneth formula

$$0 \to \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^n(X \times Y) \xrightarrow{\beta} \bigoplus_{i+j=n+1} H^i(X) * H^j(Y) \to 0$$

Given $u \in H^i(X)$ define $u^\bullet \in H^i(X \times Y)$ by $u^\bullet = p_X^*(u)$ where $p_X \colon X \times Y \to X$ is the projection. For $v \in H^j(Y)$ define $v^\bullet \in H^j(X \times Y)$ similarly and recall $u \times v = u^\bullet \cup v^\bullet$ where $\cup$ denotes the cup product.

Theorem 9.1. *With notation as above and non-zero $m \in \mathbb{Z}$*

$$\langle\!\langle u, m, v \rangle\!\rangle = \langle u^\bullet, (m)^\bullet, v^\bullet \rangle$$

where $\langle u^\bullet, (m)^\bullet, v^\bullet \rangle$ is the Massey triple product of the indicated cohomology classes where $(m)^\bullet$ is m times the multiplicative identity in $H^0(X \times Y)$.

The proof is immediate from Formula 2.1 and the definition of the Massey triple product.

10. Weakly split chain complexes

Heller's category in [3] carries much the same information as weak splittings.

Proposition 10.1. *If* $e_*\colon A_* \to B_*$ *and* $f_*\colon B_* \to C_*$ *are weakly split chain maps, then* $f_* \circ e_*$ *is weakly split by* $\mathcal{Z}_n^{f_*\circ e_*} = \mathcal{Z}_n^{f_*} \circ \mathcal{Z}_n^{e_*}$ *and* $\Psi_n^{f_*\circ e_*} = f_{n+1} \circ \Psi_n^{e_*} + \Psi_n^{f_*} \circ \mathcal{Z}_n^{e_*}$. *With these choices*

$$U\langle r\rangle_n^{f_*\circ e_*} = (f_{n+1} \otimes 1_{R/(r)}) \circ U\langle r\rangle_n^{e_*} + U\langle r\rangle_n^{f_*} \circ e_n$$

Proof. Formula (6.1) is immediate. Formula (6.2) is a routine calculation. It is straightforward to check $\Phi_*^{f_*\circ e_*} = f_{n+1} \circ \Phi_n^{e_*} + \Phi_n^{f_*} \circ \mathcal{B}_n^{e_*}$ from which the formula for the $U\langle r\rangle_*$ follows. □

Remark 10.2. Composition can be checked to be associative. The pair $\mathcal{Z}_*^{1_{A_*}} = 1_{A_*}$ and $\Psi_*^{1_{A_*}} = 0$ give the identity for any weak spitting of A_*. Hence weakly split chain complexes and weakly split chain maps form a category.

Proposition 10.3. *Let* $e_*\colon A_* \to B_*$ *be a weakly split chain map and suppose* $f_*\colon A_* \to B_*$ *is a chain map chain homotopic to* e_*. *Let* $D_*\colon A_* \to B_{*+1}$ *be a chain homotopy with*

$$f_* - e_* = \partial_{*+1}^B \circ D_* + D_{*-1} \circ \partial_*^A$$

Then f_* *is weakly split by* $\mathcal{Z}_n^{f_*} = \mathcal{Z}_n^{e_*}$ *and*

$$\Psi_n^{f_*} = \Psi_*^{e_*} + D_n \circ \gamma_n^A + \partial_{n+2}^B \circ D_{n+1} \circ \theta_n^A$$

With these choices, $U\langle r\rangle_n^{f_*} = U\langle r\rangle_n^{e_*}$

Proof. Since chain homotopic maps induce the same map in homology, it is possible to take $\mathcal{Z}_n^{e_*} = \mathcal{Z}_n^{f_*}$ and then $\mathcal{B}_n^{e_*} = \mathcal{B}_n^{f_*}$ The required verifications are straightforward. □

The remaining results are routine verifications.

Proposition 10.4. *Given two weakly split chain complexes,* $\{A_*, \mathfrak{g}_*^A\}$ *and* $\{C_*, \mathfrak{g}_*^C\}$, *then* $A_* \oplus B_*$ *is weakly split by the following data:* $\mathcal{Z}_n^{A\oplus B} = \mathcal{Z}_n^A \oplus \mathcal{Z}_n^C$, $\gamma_n^{A\oplus B} = \gamma_n^A \oplus \gamma_n^C$. *Then* $\mathcal{B}_n^{A\oplus B} = \mathcal{B}_n^A \oplus \mathcal{B}_n^C$ *so let* $\theta_n^{A\oplus B} = \theta_n^A \oplus \theta_n^C$.

Proposition 10.5. *Given weakly split chain maps* $e_*\colon A_* \to C_*$ *and* $f_*\colon B_* \to D_*$ *then* $e_* \oplus f_*$ *is weakly split by* $\mathcal{Z}_n^{e_*\oplus f_*} = \mathcal{Z}_n^{f_*} \oplus \mathcal{Z}_n^{e_*}$ *and* $\Psi_n^{e_*\oplus f_*} = \Psi_n^{e_*} \oplus \Psi_n^{f_*}$. *With these choices*

$$U\langle r\rangle_n^{e_*\oplus f_*} = U\langle r\rangle_n^{e_*} \oplus U\langle r\rangle_n^{f_*}$$

Remark 10.6. The zero complex with its evident splitting is a zero for the direct sum operation. The zero chain map between any two weakly split complexes is weakly split by letting $\mathcal{Z}_n^{0_*}$ and $\Psi_n^{0_*}$ be trivial. Then $U\langle r\rangle_n^{0_*}$ is also trivial.

There is an internal sum result.

Proposition 10.7. *Given weakly split chain maps* $e_*\colon A_* \to C_*$ *and* $f_*\colon A_* \to C_*$ *then* $e_* + f_*$ *is weakly split by* $\mathcal{Z}_n^{e_*+f_*} = \mathcal{Z}_n^{f_*} + \mathcal{Z}_n^{e_*}$ *and* $\Psi_n^{e_*+f_*} = \Psi_n^{e_*} + \Psi_n^{f_*}$. *With these choices*

$$U\langle r\rangle_n^{e_*+f_*} = U\langle r\rangle_n^{e_*} + U\langle r\rangle_n^{f_*}$$

Remark 10.8. Unlike the direct sum case (10.4), there does not seem to be an easy way to weakly split the tensor product.

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